Heavy-traffic Delay Optimality in Pull-based Load Balancing Systems: Necessary and Sufficient Conditions

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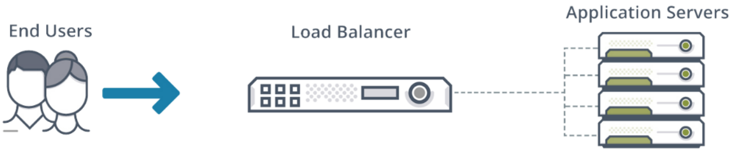
Joint work with...

Jian Tan, Alibaba

Ness Shroff, OSU

The Model...



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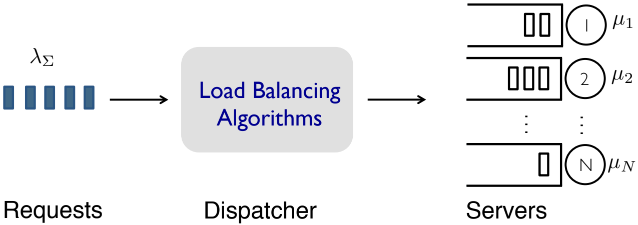
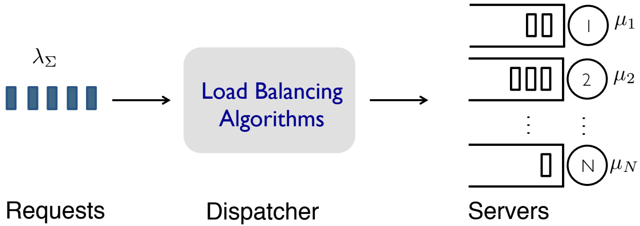


λ_Σ
Load Balancing Algorithms
1
μ_1
2
μ_2
N
μ_N
Requests
Dispatcher
Servers

The Model...



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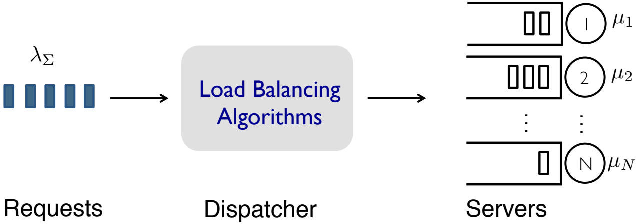

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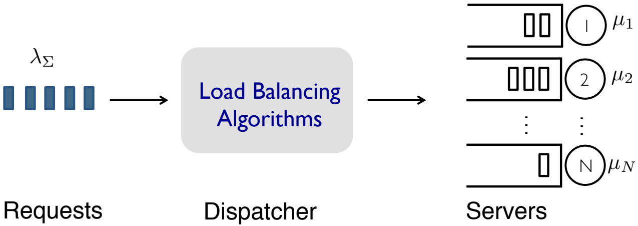

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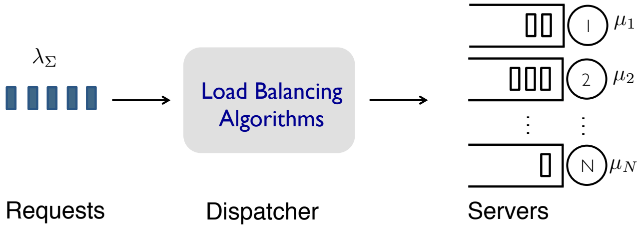

- Discrete-time system, i.e, time-slotted.
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- Arrival and service are independent.

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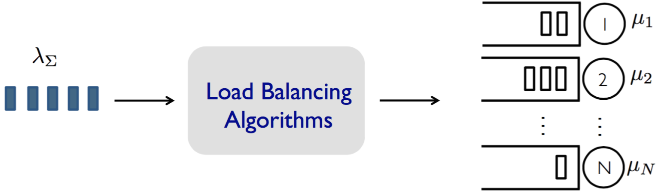

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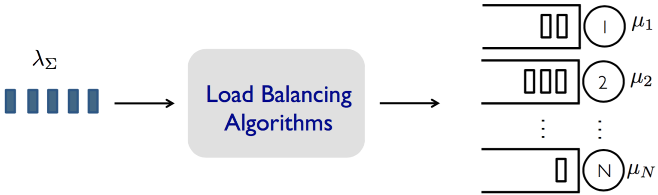

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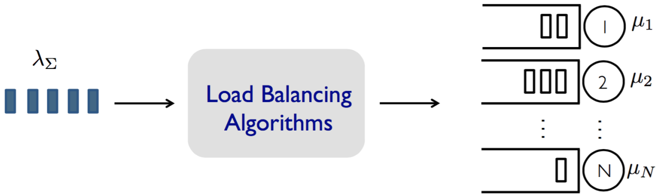

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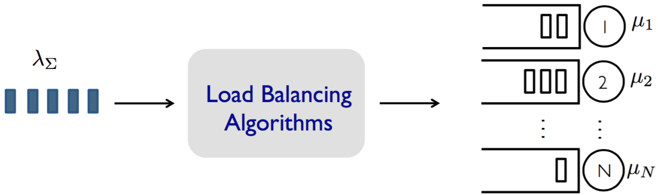

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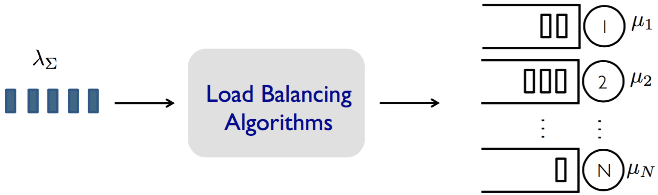

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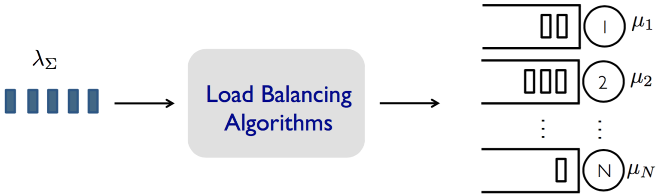

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- Low latency
  1. dispatching time (arrive at dispatcher – joining a server).
  2. response time (joining a server – leaving the server).
     - by Little's law, minimize the mean number of requests in system.

Which load balancing policy is the best?

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Maybe the most intuitive one: Join the Shortest Queue

Looking back...

Latency $=$ dispatching time $+$ response time (delay).

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- **From 1993...**
  - Logarithmic growth rate of threshold guarantees heavy-traffic delay optimality in several scheduling settings [Harrison'98], [Williams, et al'01].
  - However, it is still open in the load balancing setting.

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    - The results are more general compared to Kelly's original conjecture.
  - Provides new **insights** into heavy-traffic delay optimality.
    - The 'King' equation.
  - Develops new **techniques** for the analysis of load balancing policies.
    - New type of *state-space collapse*.

Queueing Systems 13 (1993) 47–86 47

Dynamic routing in open queueing networks: Brownian models, cut constraints and resource pooling

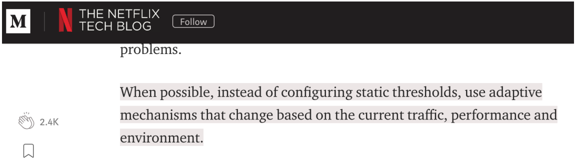
F.P. Kelly and C.N. Laws*

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

Contributions...

The necessary and sufficient conditions on the threshold have...

- Practical contributions:
  - Provides a simple **guideline** for practical systems, e.g., Netflix Zuul.
  - Sheds light on the **design** of new pull-based algorithms.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problems.

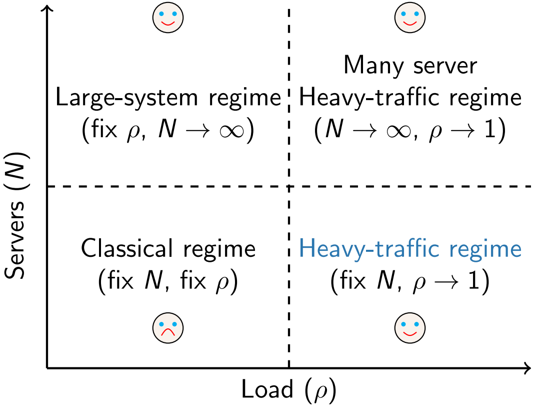
When possible, instead of configuring static thresholds, use adaptive mechanisms that change based on the current traffic, performance and environment.

2.4K

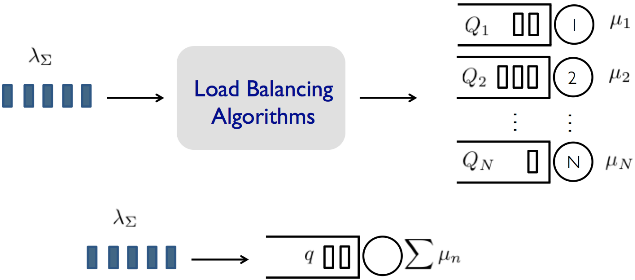
Part I: Background

Low delay...

- Closed-form formula in classical regime is very difficult.
- Turn to asymptotic regimes for insights.



Heavy-traffic Delay Optimality

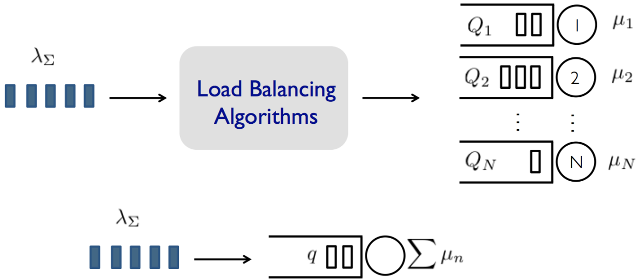


Fact: $\mathbb{E}\left[\sum Q_n\right] \geq \mathbb{E}\left[q\right]$, since one service process is stochastically larger.

Heavy-traffic Delay Optimality

Definition

It can achieve the lower bound on delay when $\epsilon \to 0$ ($\epsilon = \sum \mu_n - \lambda_\Sigma$), that is, $\lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[\sum Q_n\right] = \lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[q\right]$ (the queue length is on the order $O(1/\epsilon)$)



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The hope is that:

- instead of only storing idle servers.
- the dispatcher stores servers with queue lengths being **less than** a dynamic threshold!

Part II: Necessary Conditions

The general pull-based policy...

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- For heterogeneous servers, we can just replace random selection with selection in proportion to the service rate.

Geometry of JBT...

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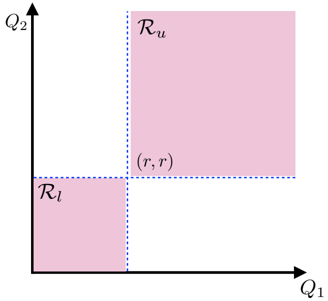

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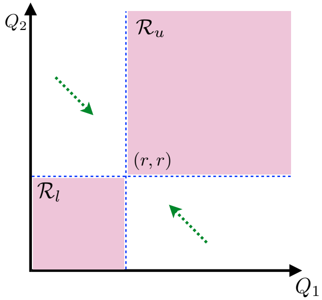

$\mathbf{Q} \notin \mathcal{R}_l \cup \mathcal{R}_u$: shorter queues are preferred.

Grow, but not too fast...

Theorem (Necessary Conditions)

Consider the JBT(r) policy.

1. *For any constant threshold r, we have the following average delay ordering in heavy traffic:*

$$\overline{D}_{JSQ} < \overline{D}_{JBT(r)} < \overline{D}_{Rand}$$

2. *For $r = \Omega((1/\epsilon)^{1+\alpha})$ for any $\alpha > 0$, we have that in heavy traffic:*

$$\overline{D}_{JSQ} < \overline{D}_{JBT(r)} = \overline{D}_{Rand}$$

Before the proof, any intuitions?

The 'King' equation...

The sufficient and necessary condition for HT-optimality:

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\left\| \overline{\mathbf{Q}}^{(\epsilon)}(t+1) \right\|_1 \left\| \overline{\mathbf{U}}^{(\epsilon)}(t) \right\|_1 \right] = 0.$$

where the unused service vector $\mathbf{U}(t) = \max\{\mathbf{S}(t) - \mathbf{Q}(t) - \mathbf{A}(t), \mathbf{0}\}$.

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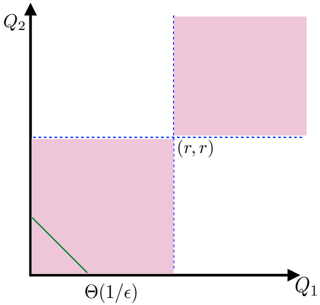
"*Probability theory is nothing but common sense reduced to calculation.*"

— Pierre Laplace



$(a) \quad r = \Omega((1/\epsilon)^{1+\alpha}), \alpha > 0$

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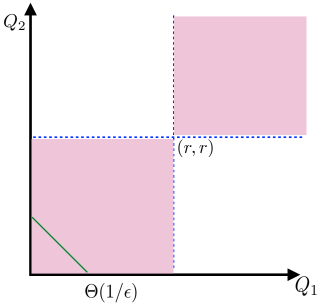

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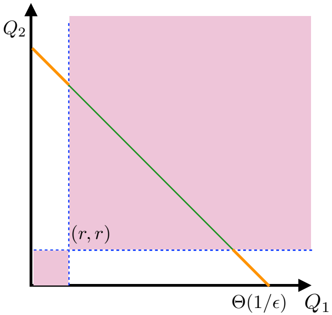

$(b) \quad r$ is a constant in $[1, \infty)$

Completely degenerate to random.

Always has 'nice' things happen.

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The Universal Equality...

$$\mathbb{E}\left[\left\|\overline{\mathbf{Q}}^{(\epsilon)}(t+1)\right\|_1\left\|\overline{\mathbf{U}}^{(\epsilon)}(t)\right\|_1\right] = \mathcal{T}_1^{(\epsilon)} + \mathcal{T}_2^{(\epsilon)} - \mathcal{T}_3^{(\epsilon)}$$

where

$$\mathcal{T}_1^{(\epsilon)} \triangleq 2\sum_{i=1}^{N}\sum_{j>i}^{N}\mathbb{E}\left[\left(\overline{Q}_i^{(\epsilon)} - \overline{Q}_j^{(\epsilon)}\right)\left(\overline{A}_i^{(\epsilon)} - \overline{A}_j^{(\epsilon)} - \overline{S}_i^{(\epsilon)} + \overline{S}_j^{(\epsilon)}\right)\right]$$

$$\mathcal{T}_2^{(\epsilon)} \triangleq \sum_{i=1}^{N}\sum_{j>i}^{N}\mathbb{E}\left[\left(\overline{A}_i^{(\epsilon)} - \overline{A}_j^{(\epsilon)} - \overline{S}_i^{(\epsilon)} + \overline{S}_j^{(\epsilon)}\right)^2\right]$$

$$\mathcal{T}_3^{(\epsilon)} \triangleq \sum_{i=1}^{N}\sum_{j>i}^{N}\mathbb{E}\left[\left(\overline{U}_i^{(\epsilon)} - \overline{U}_j^{(\epsilon)}\right)^2\right]$$

$$\overline{\mathbf{Q}}^{+} \triangleq \overline{\mathbf{Q}}(t+1)$$

The first unified method to show a policy is NOT optimal.

Then...how fast should it grow?

Part III: Sufficient Conditions

Conjecture time....

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(C). $r = \theta(\log^2(1/\epsilon))$
(D). $r = \theta((1/\epsilon)^{1.01})$

Hint: The average number of tasks is on the order of $1/\epsilon$.

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Conjecture time....

Consider the JBT(r) policy with load parameter $\epsilon = \sum \mu_n - \lambda_\Sigma$.

Question: Which of the following r value guarantees 'optimality'?

(A). $\cancel{r = 100}$
(B). $r = \theta\left(\log(1/\epsilon)\right)$
(C). $r = \theta(\log^2(1/\epsilon))$
(D). $\cancel{r = \theta((1/\epsilon)^{1.01})}$

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25 years ago...

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Queueing Systems 13 (1993) 47–86 47

Dynamic routing in open queueing networks: Brownian models, cut constraints and resource pooling

F.P. Kelly and C.N. Laws*

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

Conjecture: 'optimality' is guaranteed if $r \geq K \log(1/\epsilon)$.[2]

[2] Two-server case and diffusion approximation optimality

Logarithmic growth is sufficient...

Theorem (Sufficient Conditions)

Consider the JBT(r) policy with threshold r. Suppose $r \geq K \log(1/\epsilon)$ and $r = o(1/\epsilon)$, for some constant K, then it is heavy-traffic delay optimal in steady-state.

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Remark:

- This holds for any fixed finite number of servers, $2 \leq N < \infty$.
- This holds for general arrival and service distributions (with finite moment bounds).
- The optimality is directly in steady-state.

The challenge to prove it...

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(a) Diffusion approximations method.

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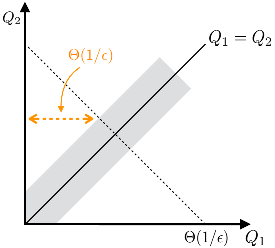
Our methods:

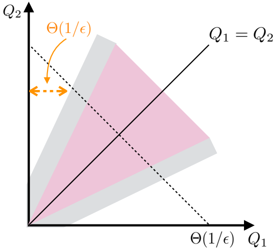
- we use drift-based analysis to establish a new type of state-space collapse.
- we combine this state-space collapse with Chernoff bound.

What is *state-space collapse?*

Informally, it means steady state *'concentrates'* around a subspace.

- the distance between the steady state and a subspace has bounded moment upper bounds.
- the subspace could be a line or a cone in previous works.





Why is *state-space collapse* important?

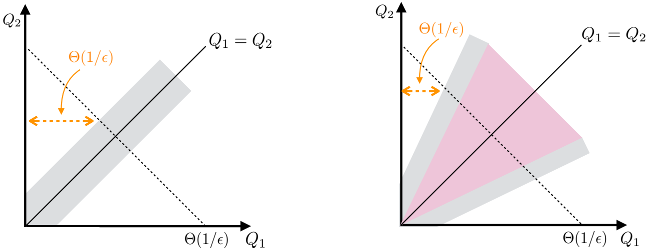
Recall the ‘King’ equation...

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\left\| \overline{\mathbf{Q}}^{(\epsilon)}(t+1) \right\|_1 \left\| \overline{\mathbf{U}}^{(\epsilon)}(t) \right\|_1 \right] = 0.$$

where the unused service vector $\mathbf{U}(t) = \max\{\mathbf{S}(t) - \mathbf{Q}(t) - \mathbf{A}(t), \mathbf{0}\}$.

- Note that $Q_n(t+1)U_n(t) = 0$ for all n and t.

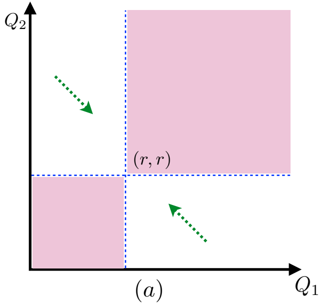
IMPLICATIONS: No server is idle while others with high loads.



In heavy traffic ($\epsilon \to 0$), steady state lies far away from boundary.

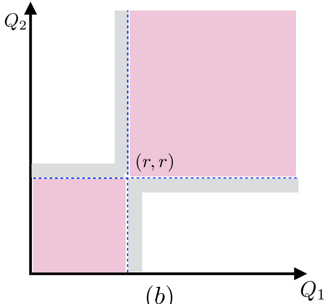
What would be the *state-space collapse* in our case?

From drift to Chernoff bound...



Drift towards the pink region due to preference of shorter queues.

From drift to Chernoff bound...



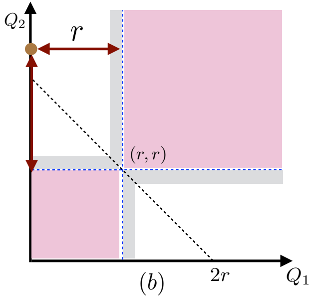
(b)

Then, ALL the moments of the distance to pink region are bounded!

$$\mathbb{E}\left[e^{\theta^* d_{\mathcal{R}(r)}\left(\overline{\mathbf{Q}}^{(\epsilon)}\right)}\right] \leq C^*,$$

where both θ^* and C^* are independent of ϵ.

From drift to Chernoff bound...

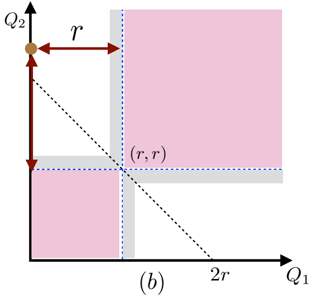


$$\mathbb{E}\left[\overline{Q}_2^{(\epsilon)}(t+1)\overline{U}_1^{(\epsilon)}\right]$$

$$=\mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1\mathcal{I}\left(\overline{Q}_2(t+1)\leq 2r,\overline{Q}_1(t+1)=0\right)\right] \quad (1)$$

$$+\mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1\mathcal{I}\left(\overline{Q}_2(t+1)> 2r,\overline{Q}_1(t+1)=0\right)\right] \quad (2)$$

From drift to Chernoff bound...



$$
\begin{aligned}
& \mathbb{E}\left[\overline{Q}_2^{(\epsilon)}(t+1)\overline{U}_1^{(\epsilon)}\right] \\
=& \mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1 \mathcal{I}\left(\overline{Q}_2(t+1) \leq 2r, \overline{Q}_1(t+1) = 0\right)\right] && (1) \\
& + \mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1 \mathcal{I}\left(\overline{Q}_2(t+1) > 2r, \overline{Q}_1(t+1) = 0\right)\right] && (2)
\end{aligned}
$$

$(1) \leq 2r\epsilon,$ since $\mathbb{E}\left[\overline{U}_1\right] \leq \epsilon.$

$$(2) \leq C\frac{1}{\epsilon^2}\mathbb{P}\left(\overline{Q}_2(t+1) > 2r, \overline{Q}_1(t+1) = 0\right) \leq C\frac{1}{\epsilon^2}\frac{1}{e^{\theta r}}$$

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The key requirement: set of servers whose queue lengths are below the threshold are known at the dispatcher

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  - If one or more IDs exist, it removes one uniformly at random, and sends all the new arrivals to the corresponding server.
  - Otherwise, all the new arrivals are dispatched uniformly at random to one of the servers in the system.
- Each server reports its ID to the dispatcher at the end of each time-slot if
  - its queue length is below the threshold
  - AND the dispatcher does not contain its ID (how?)

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Theorem

For any finite T and $d \geq 1$, the policy is throughput and delay optimal in heavy traffic.

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We conjecture so!

Conclusion...

Theorem (Necessary Conditions)

- The threshold r should grow with the traffic load.
- But, it can not grow too fast.
- It provides a sharp characterization of JIQ policy.

Theorem (Sufficient Conditions)

- It is sufficient to have a logarithmic growth rate.
- This resolves a long-standing open problem.
- It provides a useful guideline for practical systems.

Thank you!

Q & A