Heavy-traffic Delay Optimality in Pull-based Load Balancing Systems: Necessary and Sufficient Conditions

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The Ohio State University

Feb. 7 2019 @ Caltech

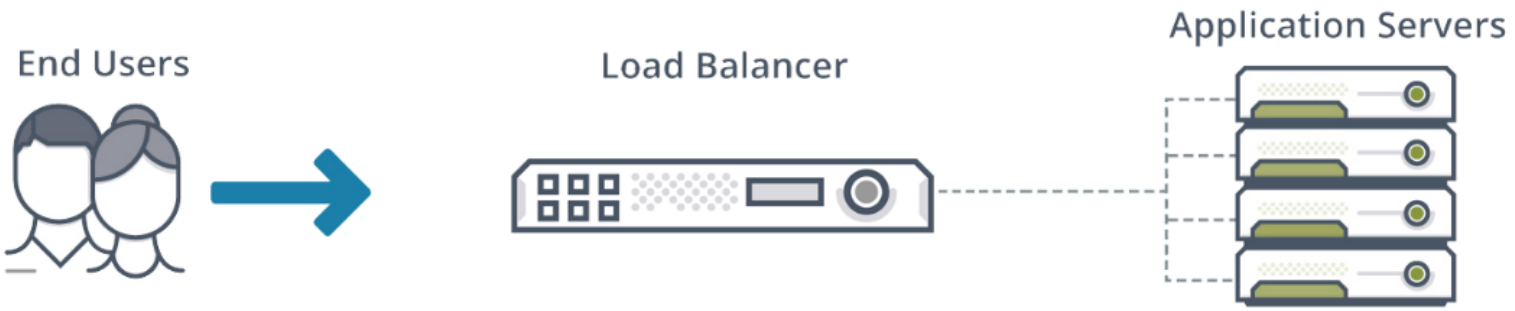
Joint work with...

Jian Tan, Alibaba

Ness Shroff, OSU

The cloud is everywhere...

Load balancing is a key...



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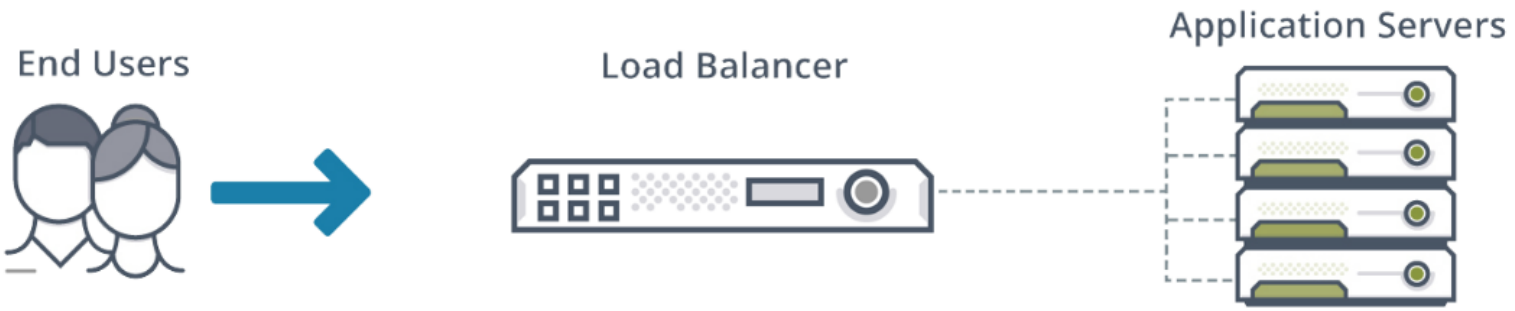

- Availability

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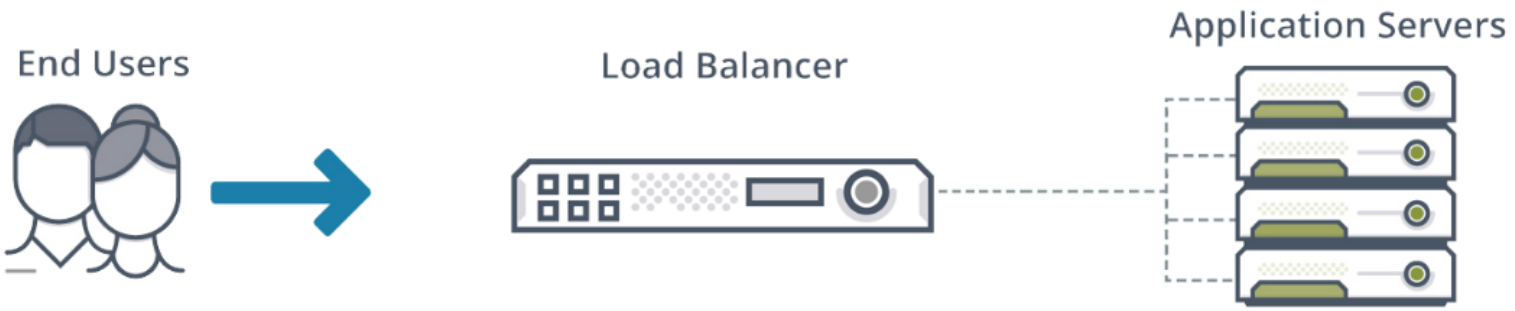
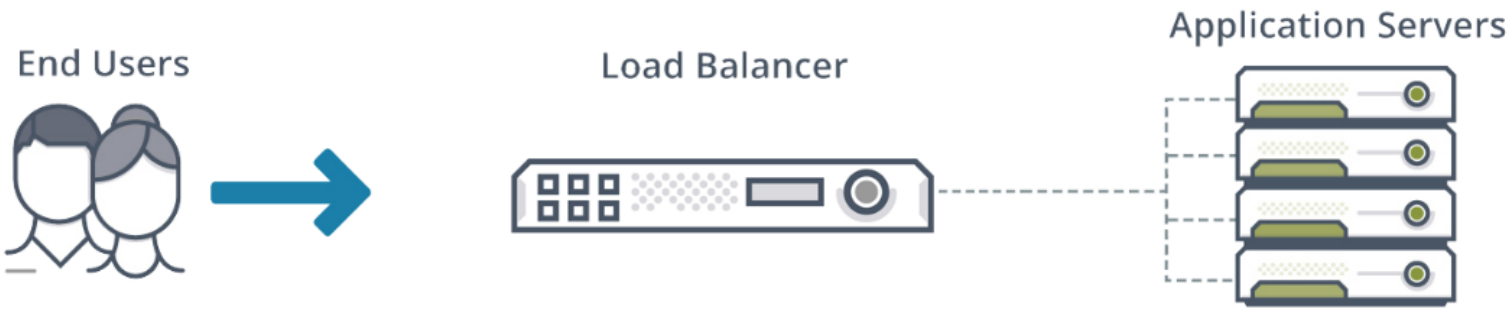


- Availability
- Throughput

Load balancing is a key...

- Availability
- Throughput
- Responsiveness
  - : a half second increase in loading time drops traffic by 20%.
  - : every 0.1s of latency costs them 1% in sales.

Let's model it...



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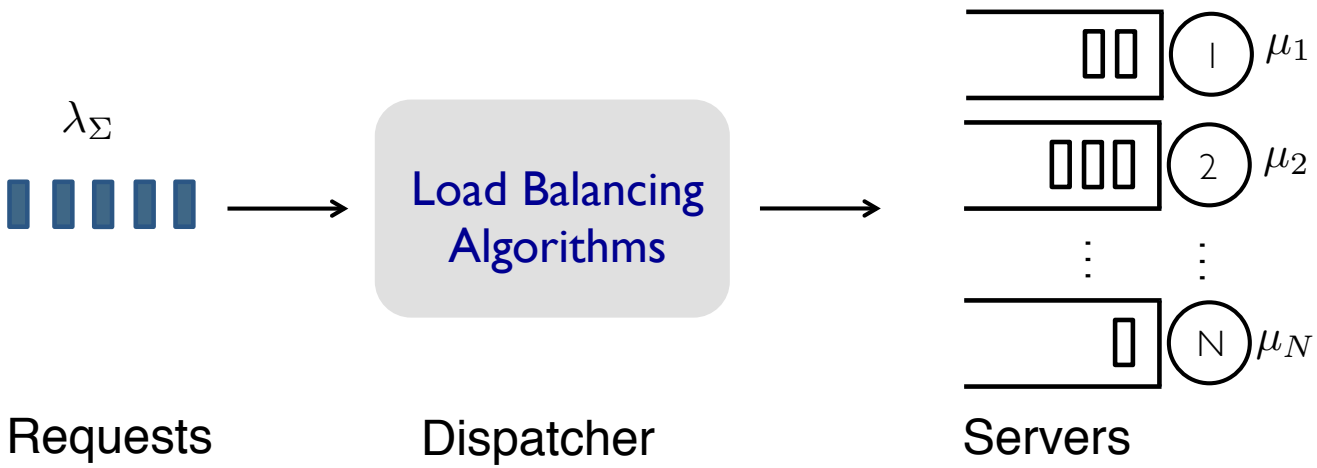

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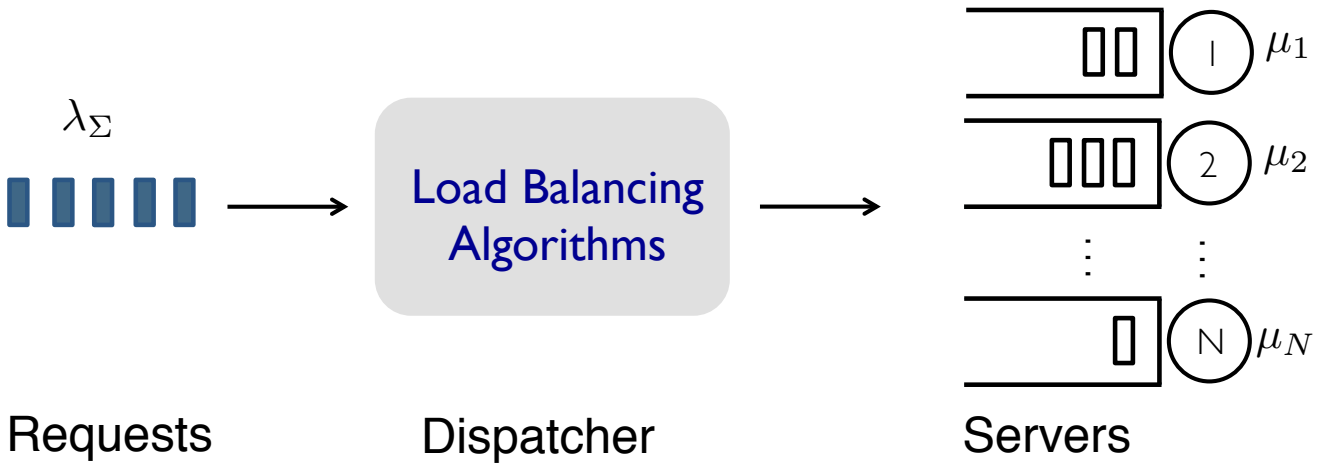

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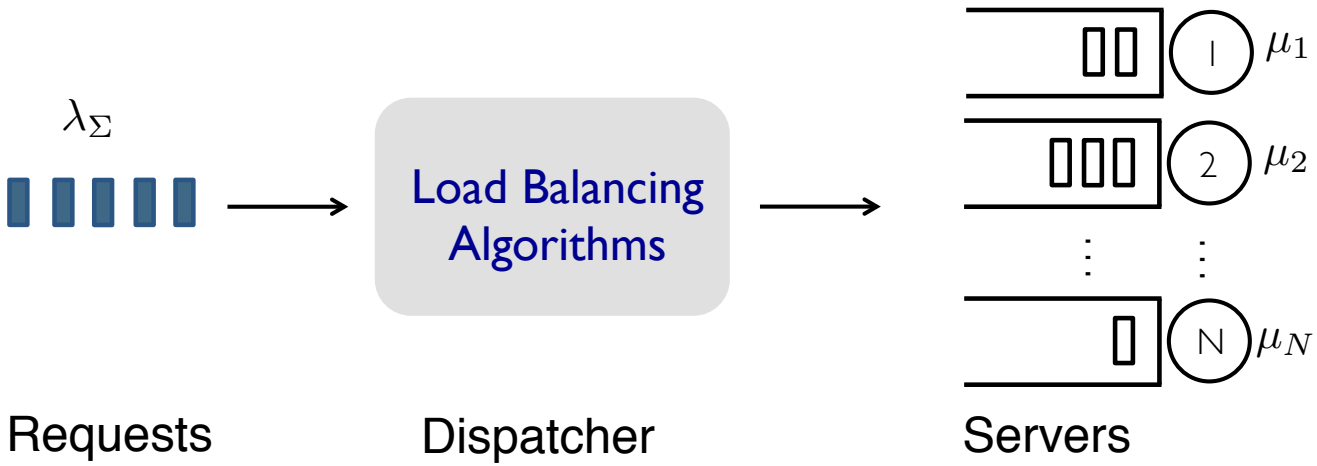

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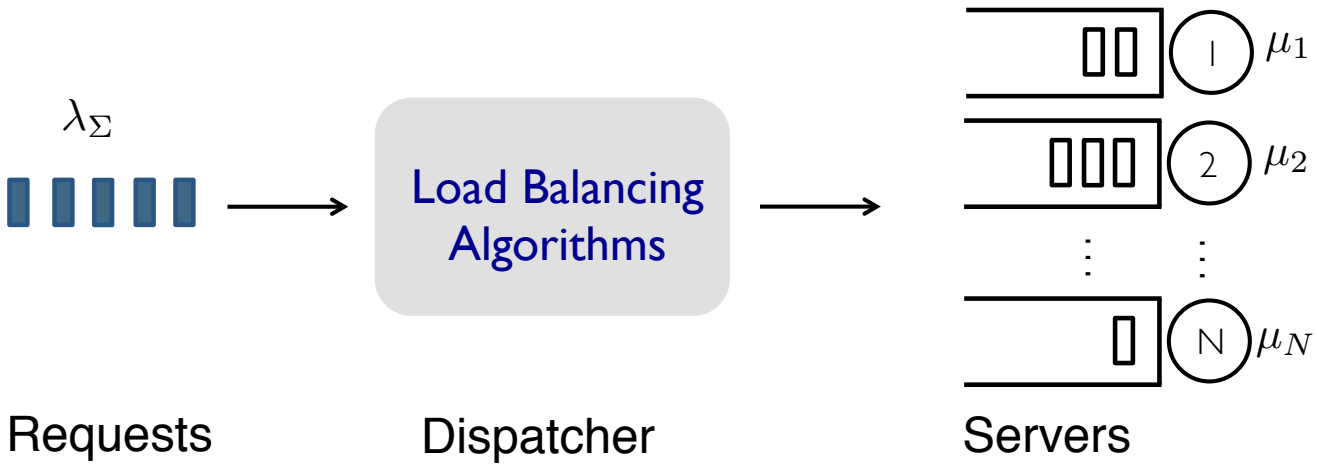

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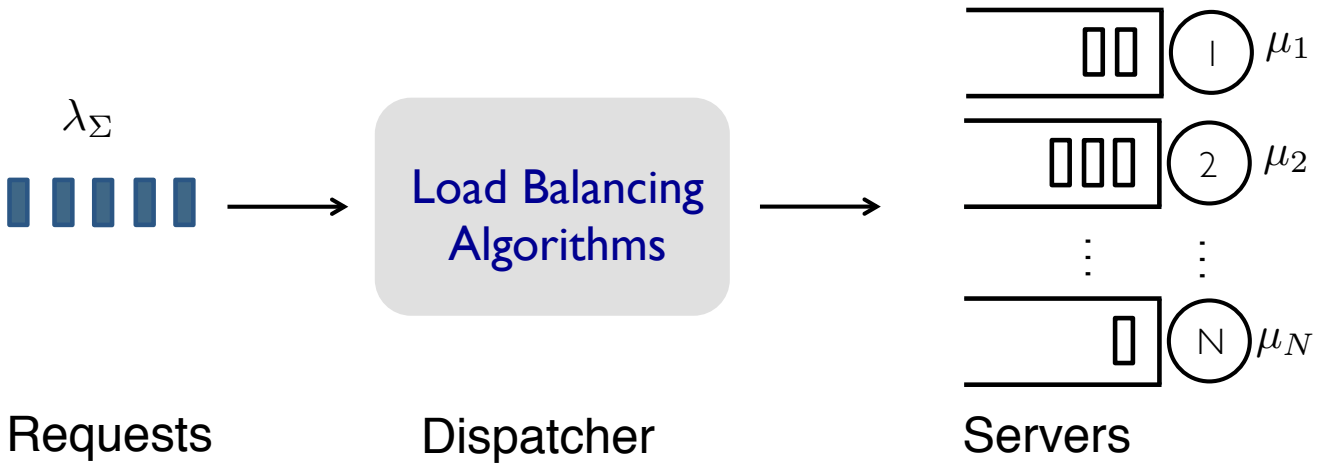

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- Arrival and service are independent.

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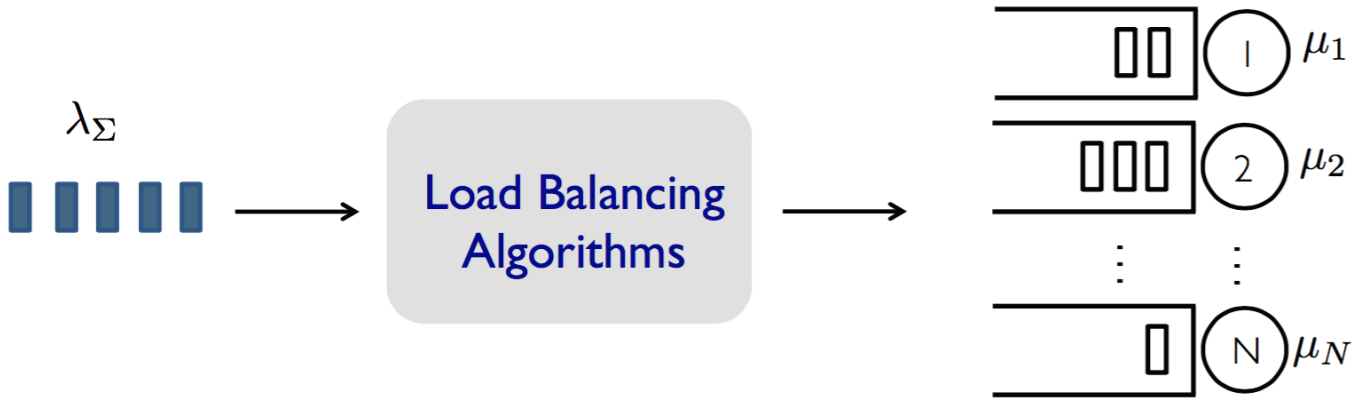

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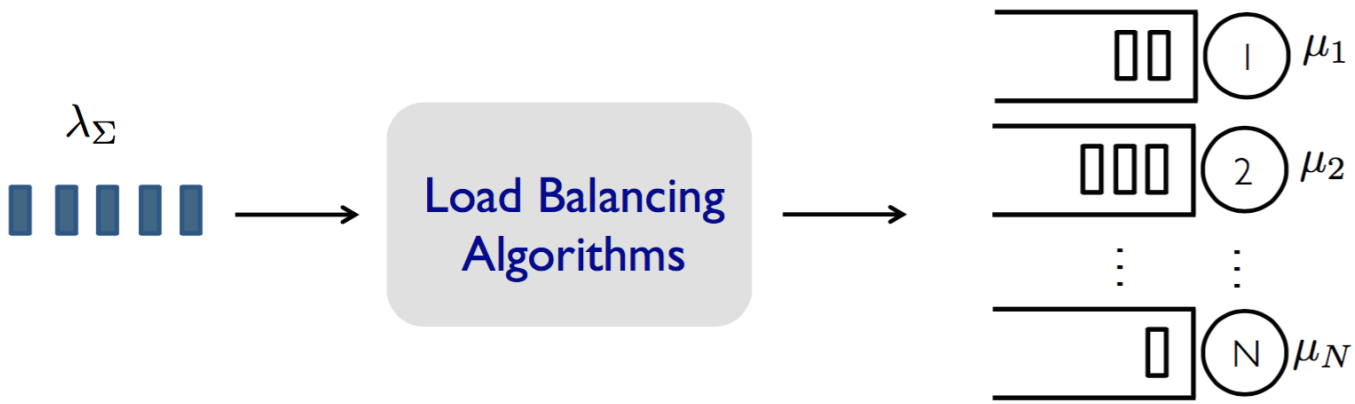

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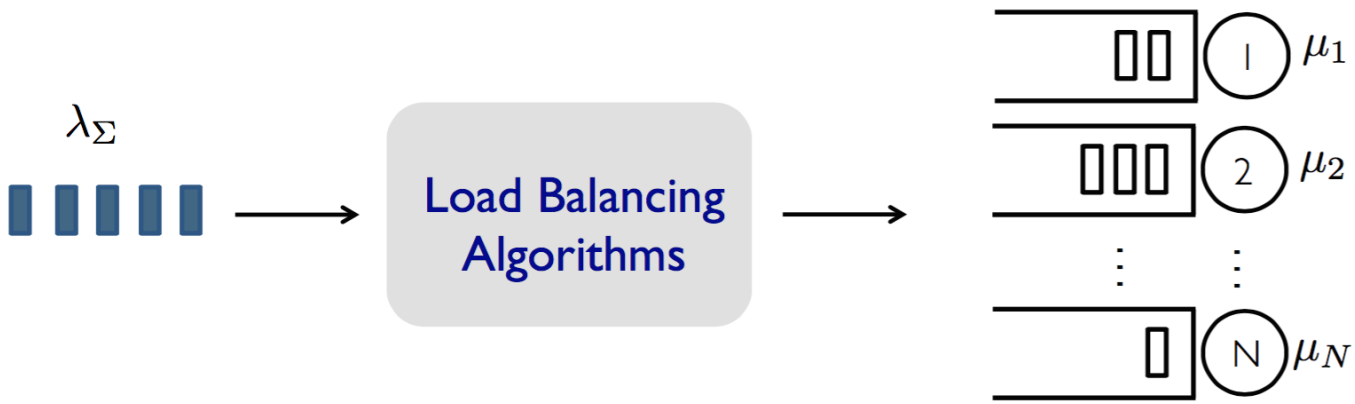

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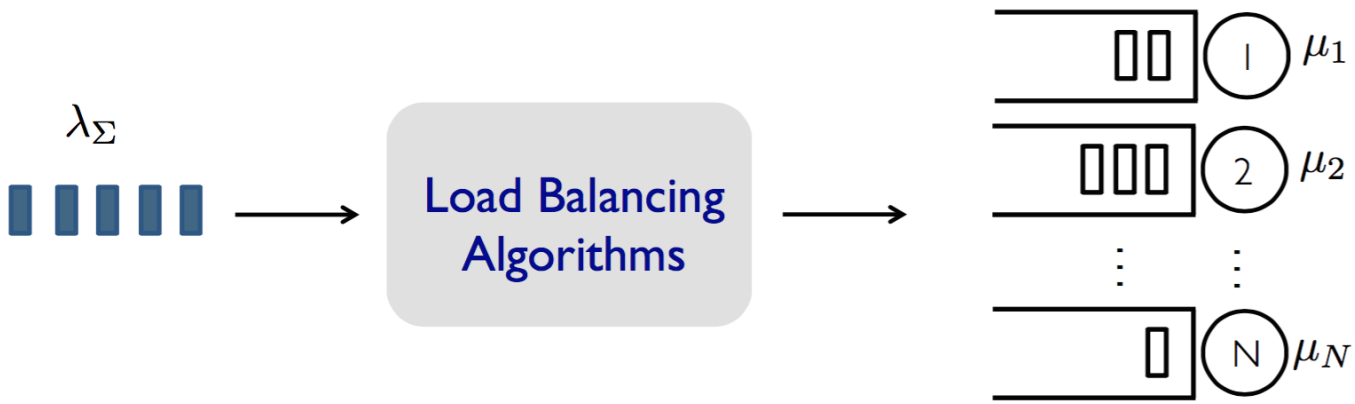

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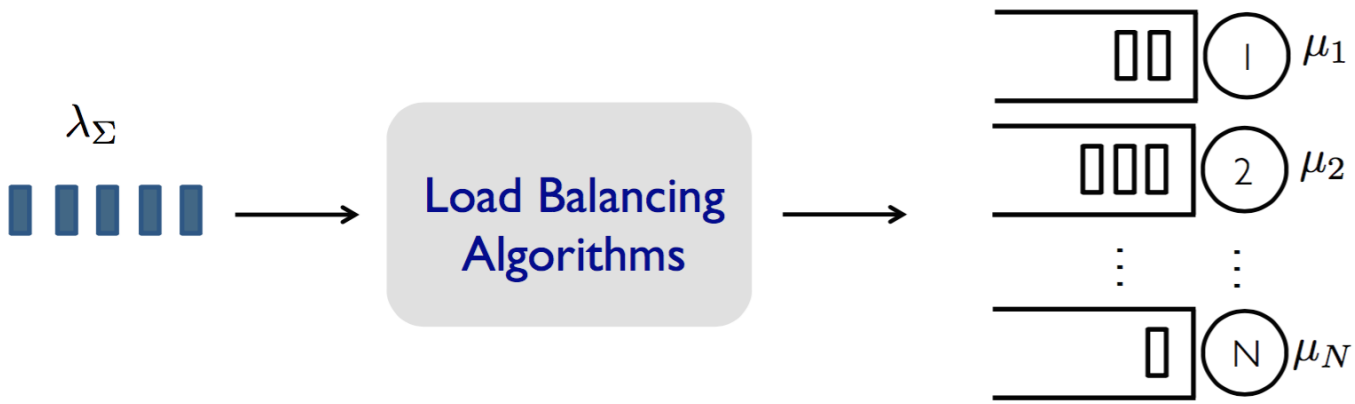

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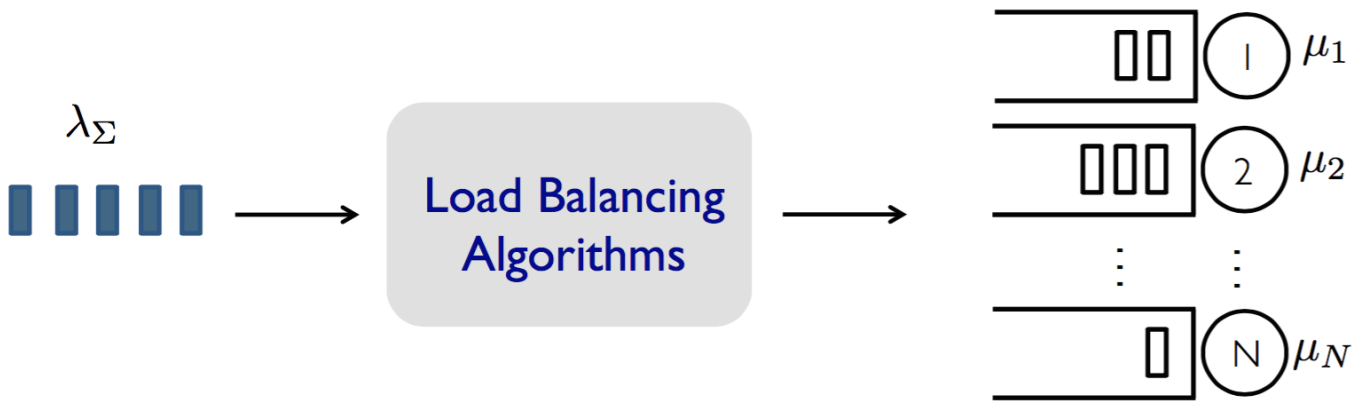

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  2. response time (joining a server – leaving the server).
     - by Little's law, minimize the mean number of requests in system.

Which load balancing policy is the best?

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Maybe the most intuitive one: Join the Shortest Queue

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- **From 1993...**
  - Logarithmic growth rate of threshold guarantees heavy-traffic delay optimality in several scheduling settings [Harrison'98], [Williams, et al'01].
  - However, it is still open in the load balancing setting.

Contributions...

We present necessary and sufficient conditions on the threshold for delay optimality in heavy traffic in **load balancing systems**.

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    - The results are more general compared to Kelly's original conjecture.
  - Provides new **insights** into heavy-traffic delay optimality.
    - The 'King' equation.
  - Develops new **techniques** for the analysis of load balancing policies.
    - New type of *state-space collapse*.

Queueing Systems 13 (1993) 47–86

Dynamic routing in open queueing networks: Brownian models, cut constraints and resource pooling

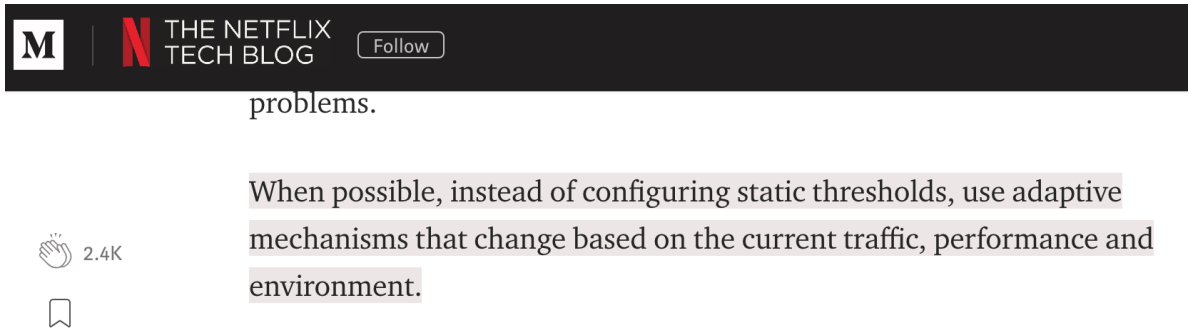
F.P. Kelly and C.N. Laws*

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

Contributions...

The necessary and sufficient conditions on the threshold have...

- Practical contributions:
  - Provides a simple **guideline** for practical systems, e.g., Netflix Zuul.
  - Sheds light on the **design** of new pull-based algorithms.



Part I: Background

Low delay...

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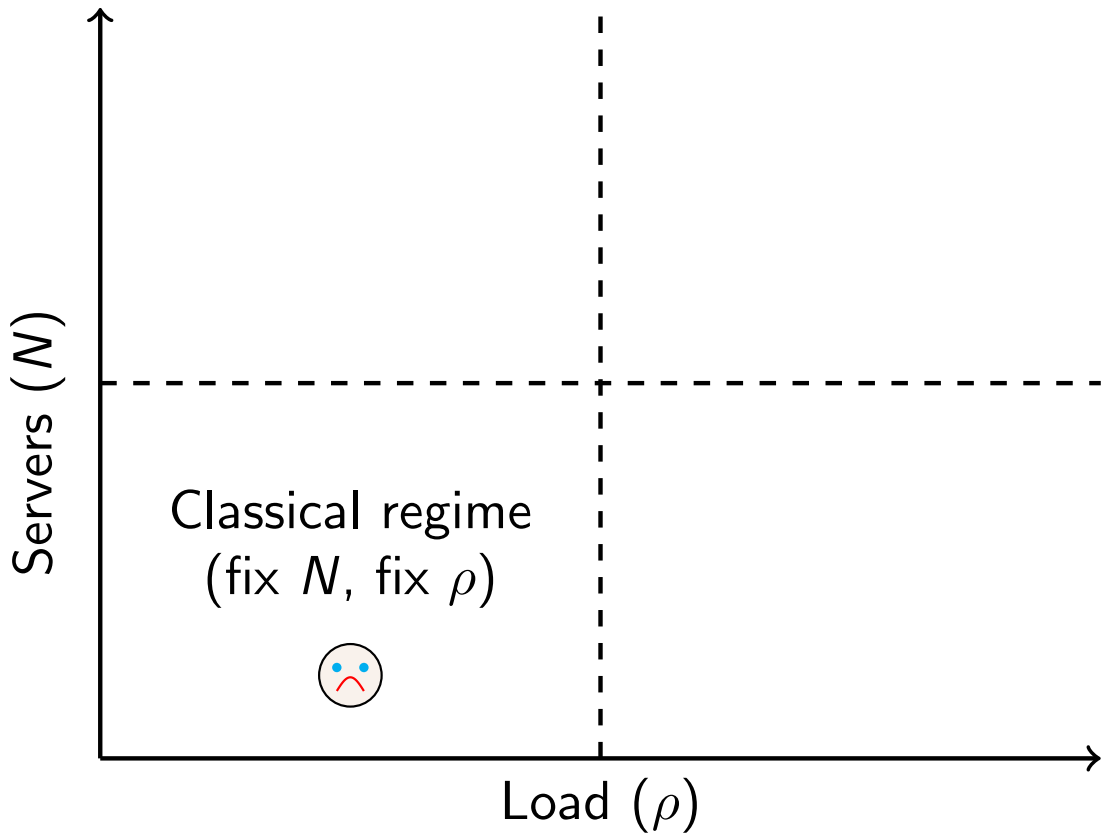
- Closed-form formula in classical regime is very difficult.



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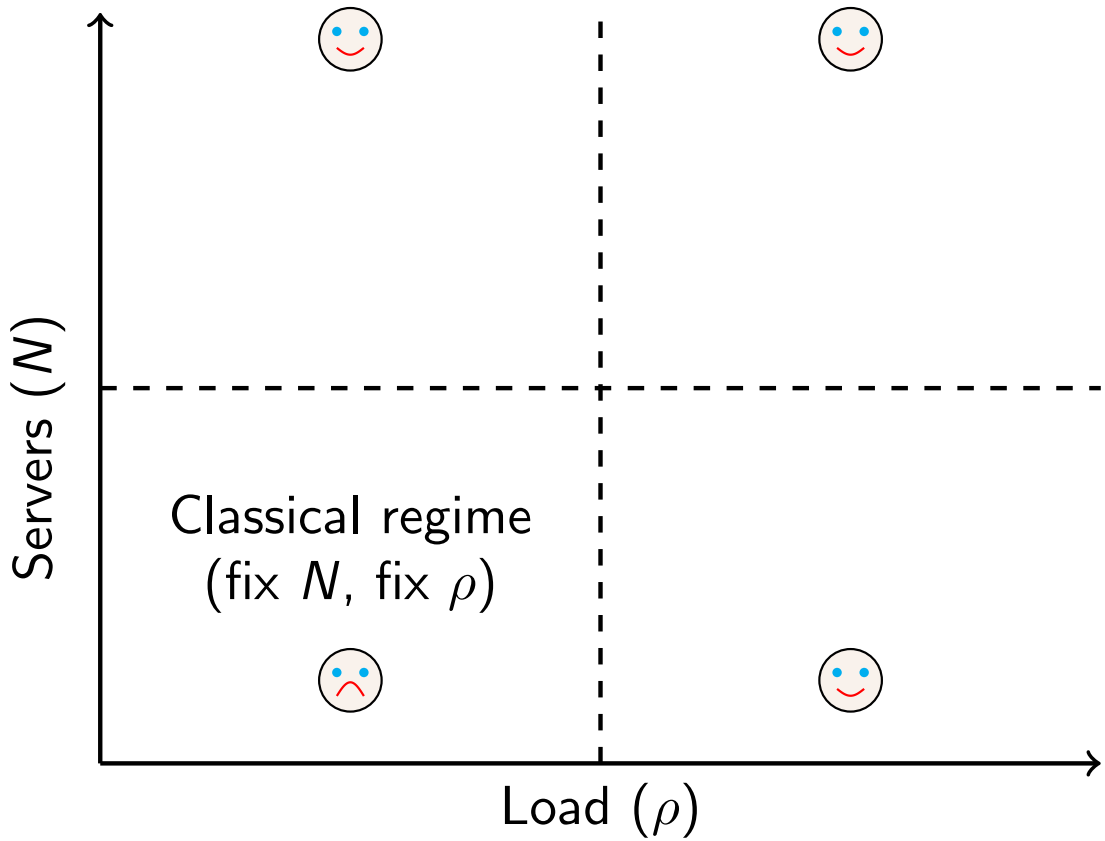

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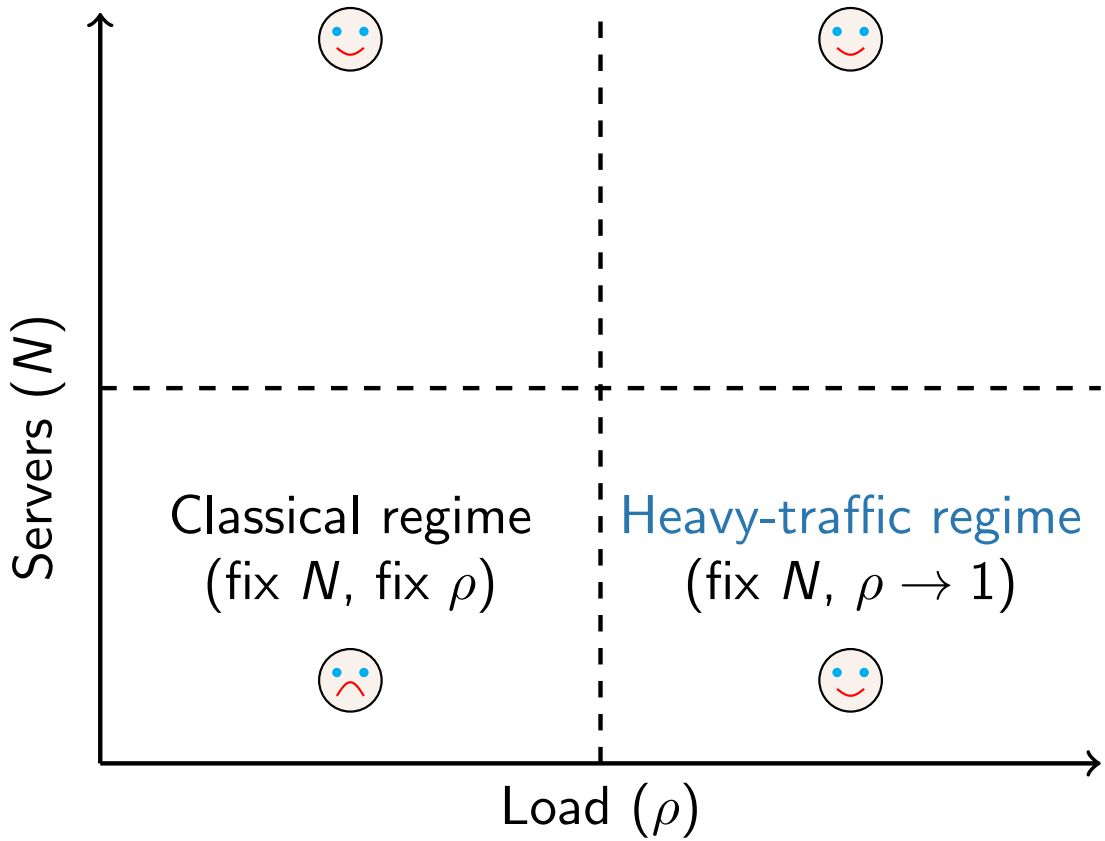

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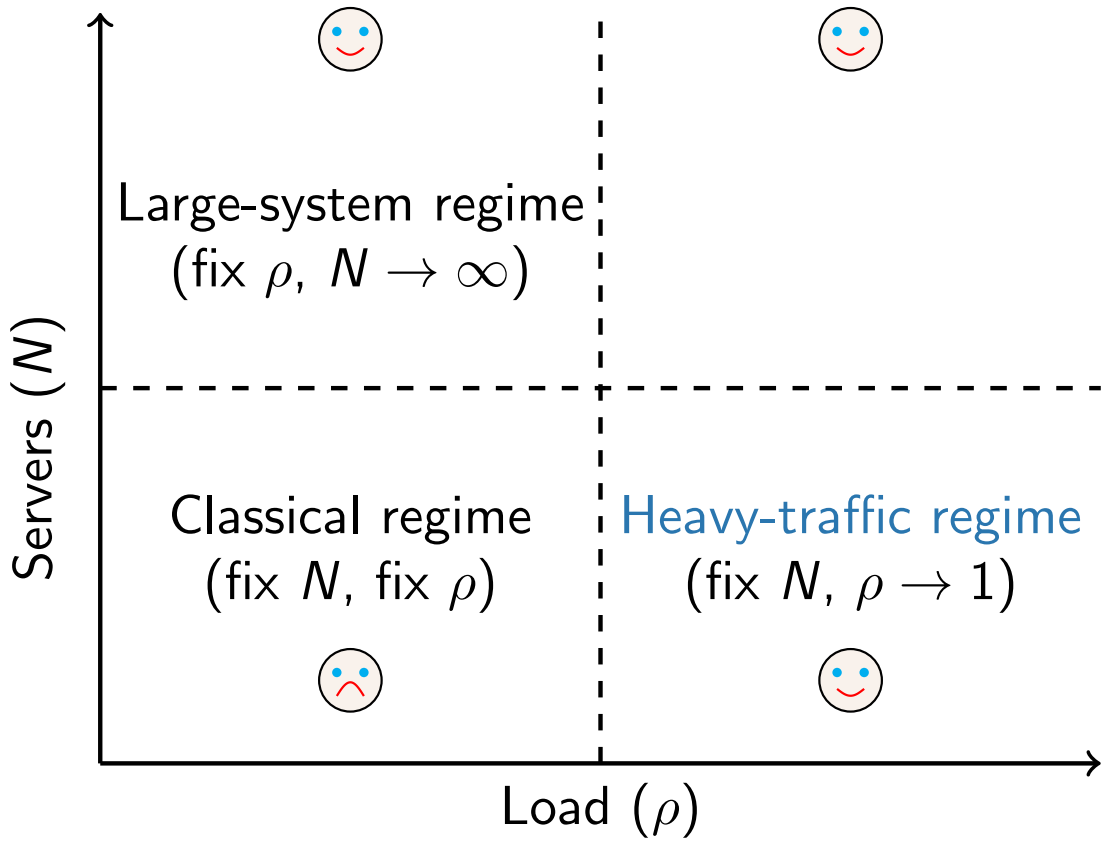

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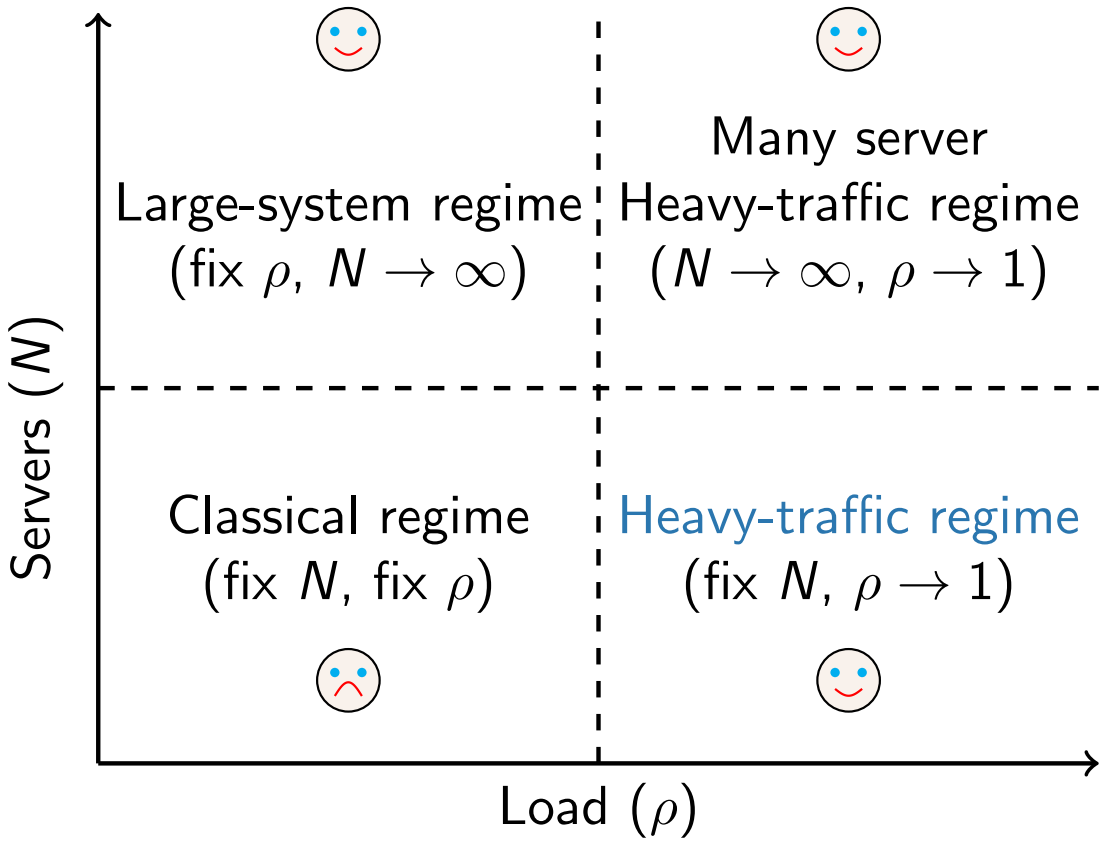

Why is the heavy-traffic regime?

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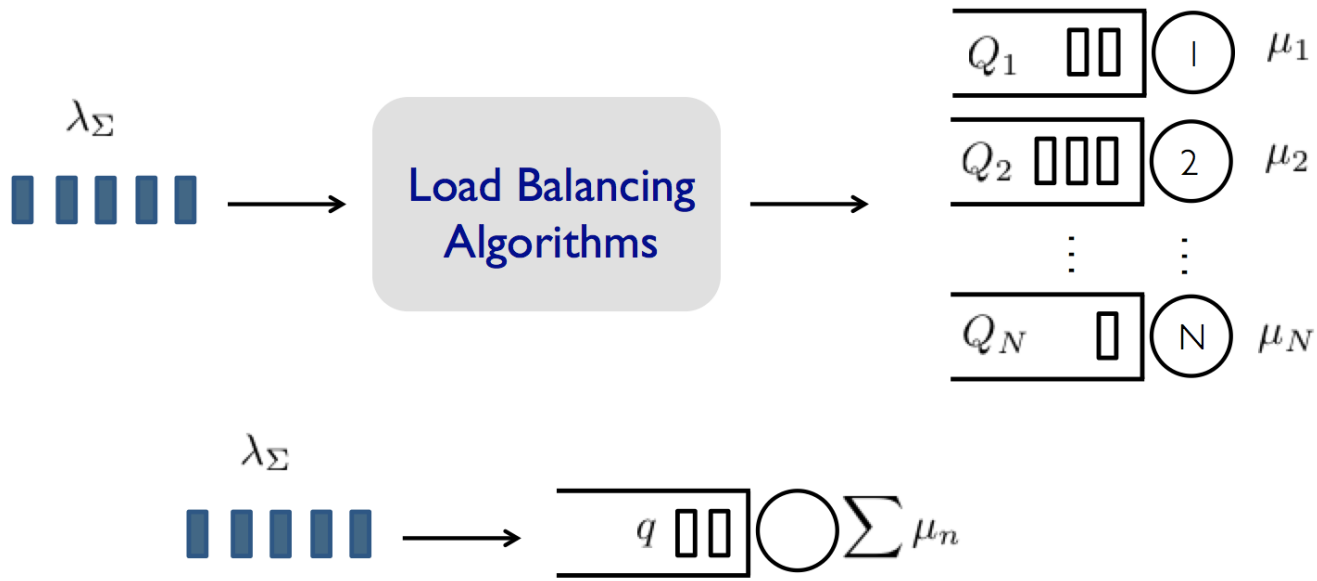
"In heavy-traffic regime, the important features of good control policies are often displayed in the sharpest relief"

— Frank Kelly

Heavy-traffic Delay Optimality

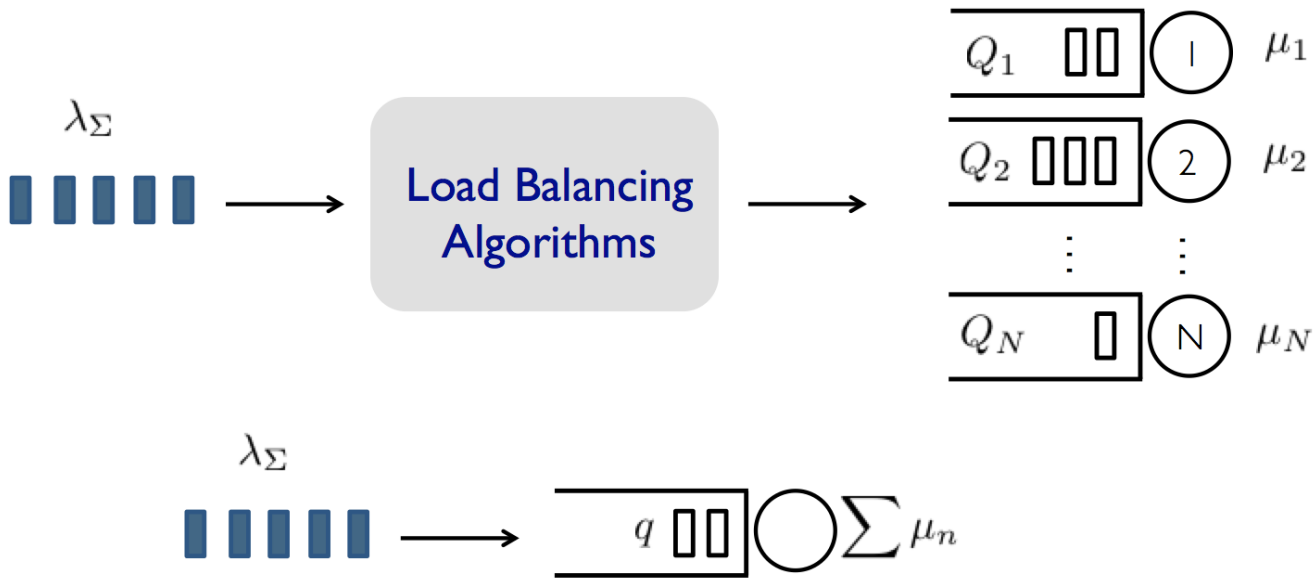


Fact: $\mathbb{E}\left[\sum Q_n\right] \geq \mathbb{E}\left[q\right]$, since one service process is stochastically larger.

Heavy-traffic Delay Optimality

Definition

It can achieve the lower bound on delay when $\epsilon \to 0$ ($\epsilon = \sum \mu_n - \lambda_\Sigma$), that is, $\lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[\sum Q_n\right] = \lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[q\right]$ (the queue length is on the order $O(1/\epsilon)$)



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- sample each queue length
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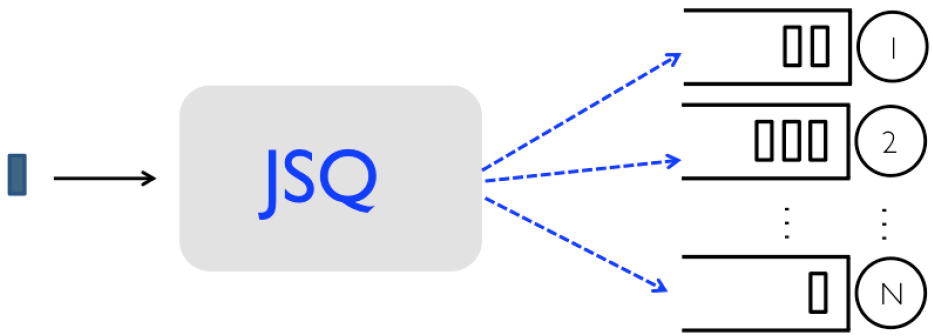

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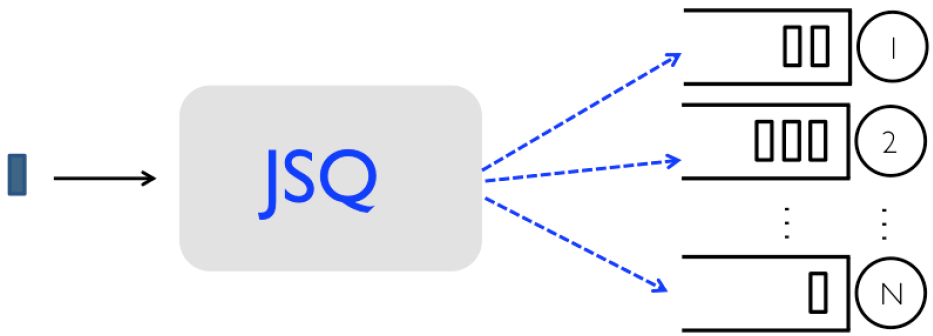

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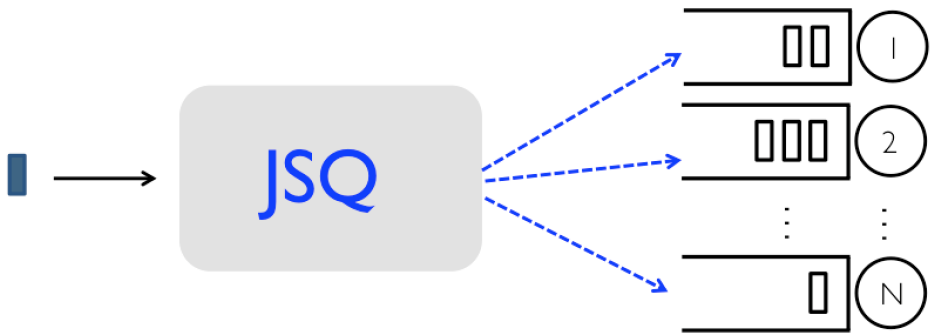

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Cons:

- Message overhead is undesriable ($2N$ per arrival).
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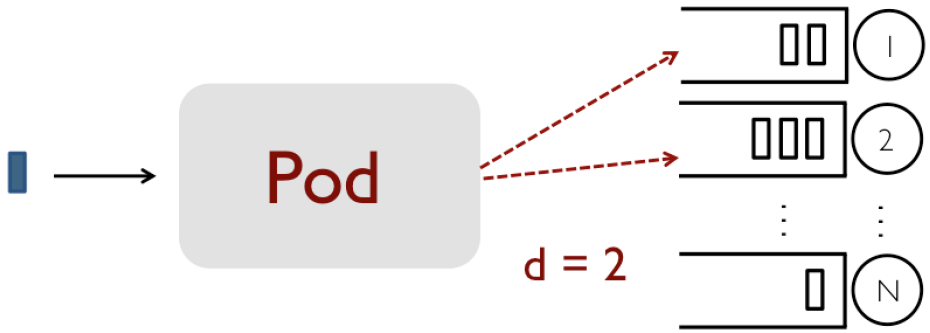

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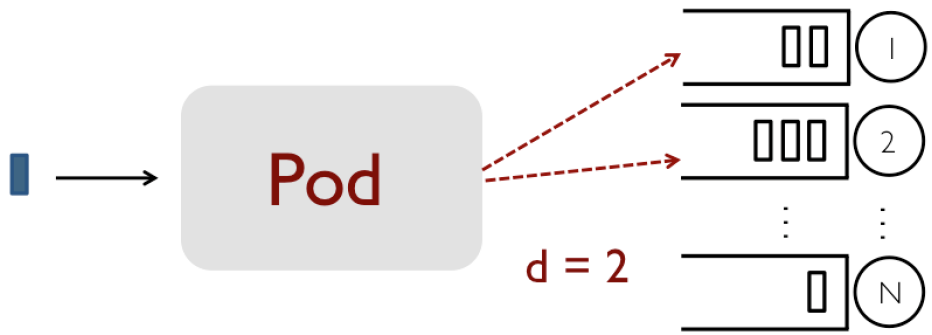

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- Double exponential decay in delay tail when N is large. [Mitzenmacher'96]
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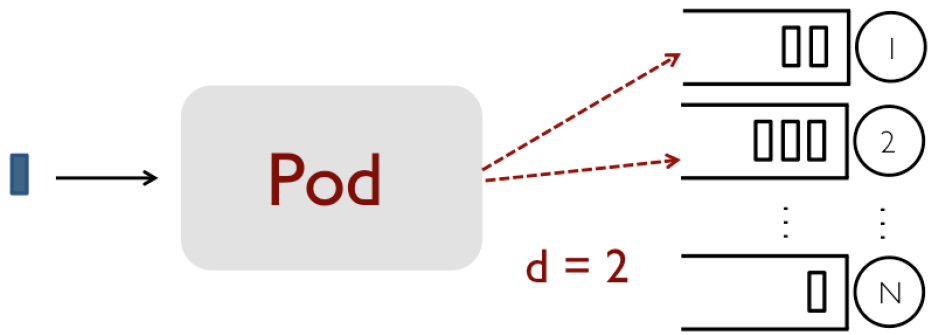

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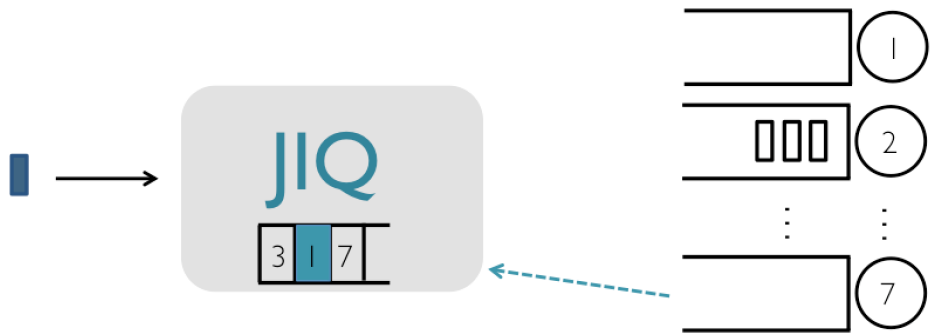

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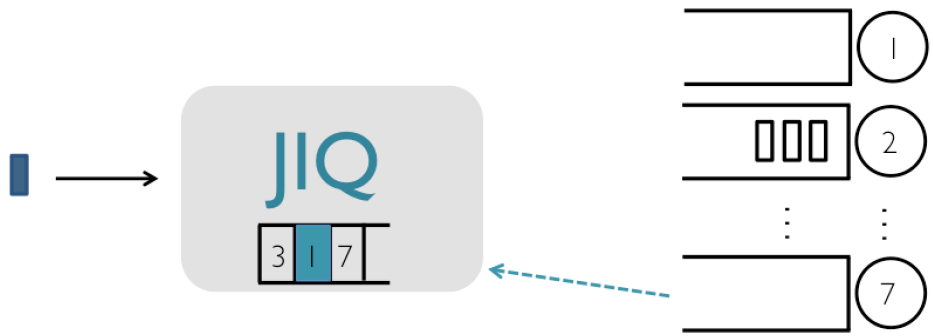

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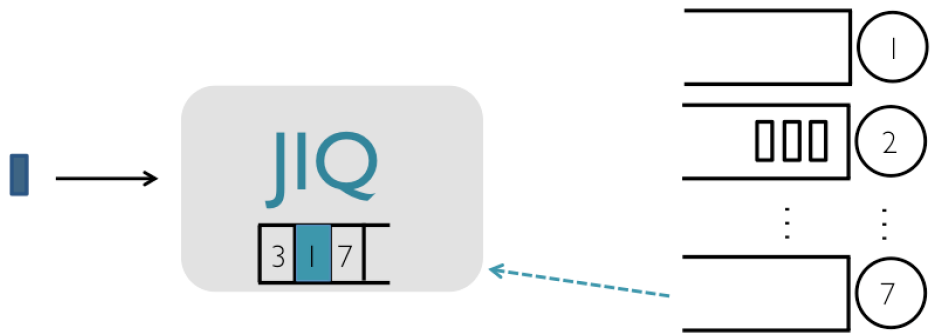

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- Push algorithms, e.g., JSQ and Pod

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Big picture...

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The hope is that:

- instead of only storing idle servers.
- the dispatcher stores servers with queue lengths being **less than** a dynamic threshold!

Part II: Necessary Conditions

Warm up....

(a) Is JIQ delay optimal in heavy traffic?

(A). Yes

(B). No

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Both answers are NO!

In fact, in the heavy-traffic limit:

heavy-traffic optimal = JSQ $<$ JIQ $<$ Rand

The general pull-based policy...

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Remark:

- JIQ is just a special case of JBT(r) with constant $r = 1$.
- For heterogeneous servers, we can just replace random selection with selection in proportion to the service rate.

Geometry of JBT...

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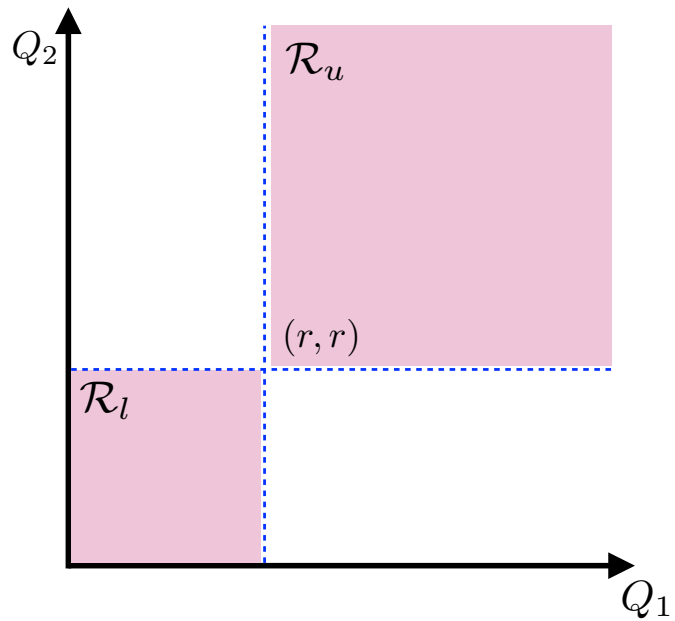

$\mathbf{Q} \in \mathcal{R}_l$: Random (full memory) $\mathbf{Q} \in \mathcal{R}_u$: Random (empty memory)

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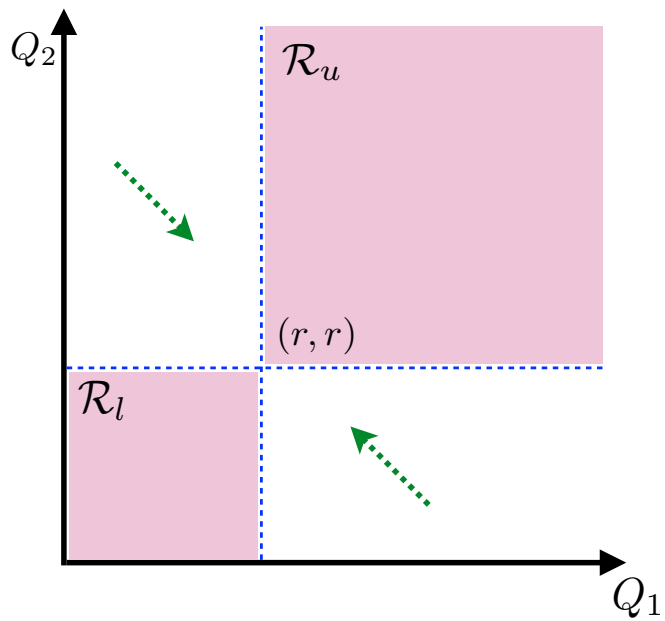

$\mathbf{Q} \notin \mathcal{R}_l \cup \mathcal{R}_u$: shorter queues are preferred.

Grow, but not too fast...

Theorem (Necessary Conditions)

Consider the JBT(r) policy.

1. *For any constant threshold r, we have the following average delay ordering in heavy traffic:*

$$\overline{D}_{JSQ} < \overline{D}_{JBT(r)} < \overline{D}_{Rand}$$

2. *For* $r = \Omega((1/\epsilon)^{1+\alpha})$ *for any* $\alpha > 0$, *we have that in heavy traffic:*

$$\overline{D}_{JSQ} < \overline{D}_{JBT(r)} = \overline{D}_{Rand}$$

Before the proof, any intuitions?

The 'King' equation...

The sufficient and necessary condition for HT-optimality:

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\left\|\overline{\mathbf{Q}}^{(\epsilon)}(t+1)\right\|_1 \left\|\overline{\mathbf{U}}^{(\epsilon)}(t)\right\|_1\right] = 0.$$

where the unused service vector $\mathbf{U}(t) = \max\{\mathbf{S}(t) - \mathbf{Q}(t) - \mathbf{A}(t), \mathbf{0}\}$.

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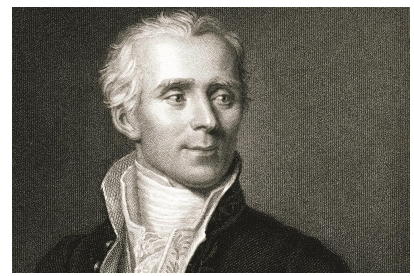
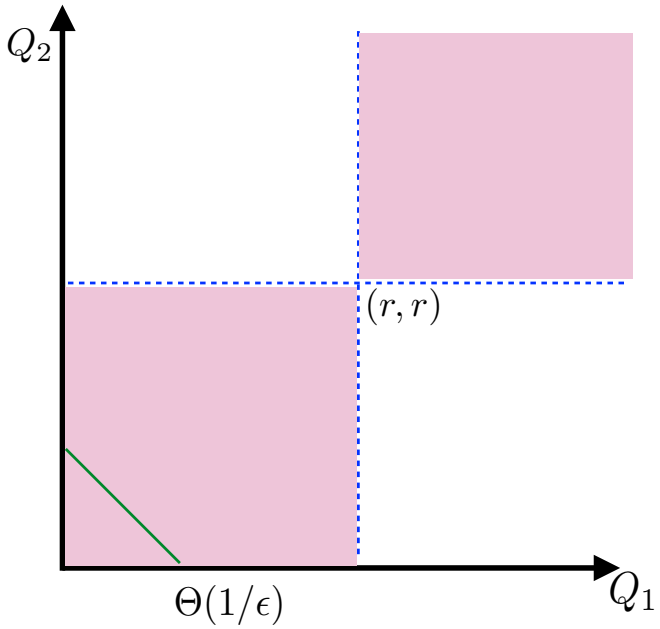
"*Probability theory is nothing but common sense reduced to calculation.*"

— Pierre Laplace





$(a) \quad r = \Omega((1/\epsilon)^{1+\alpha}), \alpha > 0$

Completely degenerate to random.

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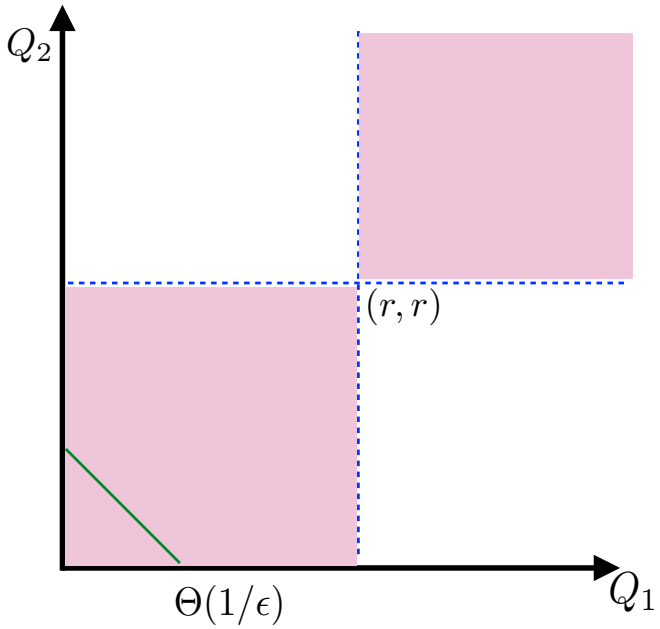

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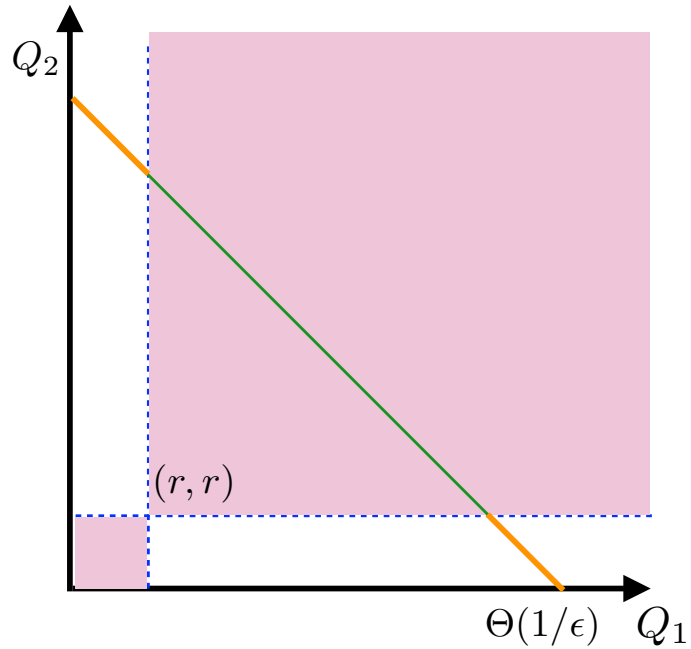

(b) r is a constant in $[1, \infty)$

Completely degenerate to random.

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Always has 'nice' things happen.

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The Universal Equality...

$$\mathbb{E}\left[\left\|\overline{\mathbf{Q}}^{(\epsilon)}(t+1)\right\|_1 \left\|\overline{\mathbf{U}}^{(\epsilon)}(t)\right\|_1\right] = \mathcal{T}_1^{(\epsilon)} + \mathcal{T}_2^{(\epsilon)} - \mathcal{T}_3^{(\epsilon)}$$

where

$$\mathcal{T}_1^{(\epsilon)} \triangleq 2\sum_{i=1}^{N}\sum_{j>i}^{N} \mathbb{E}\left[\left(\overline{Q}_i^{(\epsilon)} - \overline{Q}_j^{(\epsilon)}\right)\left(\overline{A}_i^{(\epsilon)} - \overline{A}_j^{(\epsilon)} - \overline{S}_i^{(\epsilon)} + \overline{S}_j^{(\epsilon)}\right)\right]$$

$$\mathcal{T}_2^{(\epsilon)} \triangleq \sum_{i=1}^{N}\sum_{j>i}^{N} \mathbb{E}\left[\left(\overline{A}_i^{(\epsilon)} - \overline{A}_j^{(\epsilon)} - \overline{S}_i^{(\epsilon)} + \overline{S}_j^{(\epsilon)}\right)^2\right]$$

$$\mathcal{T}_3^{(\epsilon)} \triangleq \sum_{i=1}^{N}\sum_{j>i}^{N} \mathbb{E}\left[\left(\overline{U}_i^{(\epsilon)} - \overline{U}_j^{(\epsilon)}\right)^2\right]$$

$$\overline{\mathbf{Q}}^{+} \triangleq \overline{\mathbf{Q}}(t+1)$$

The first unified method to show a policy is NOT optimal.

Then...how fast should it grow?

Part III: Sufficient Conditions

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Hint: The average number of tasks is on the order of $1/\epsilon$.

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25 years ago...

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Queueing Systems 13 (1993) 47–86

Dynamic routing in open queueing networks: Brownian models, cut constraints and resource pooling

F.P. Kelly and C.N. Laws*

Statistical Laboratory, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

Conjecture: 'optimality' is guaranteed if $r \geq K \log(1/\epsilon)$.[2]

[2] Two-server case and diffusion approximation optimality

Logarithmic growth is sufficient...

Theorem (Sufficient Conditions)

Consider the JBT(r) policy with threshold r. Suppose $r \geq K \log(1/\epsilon)$ *and* $r = o(1/\epsilon)$*, for some constant* K*, then it is heavy-traffic delay optimal in steady-state.*

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- This holds for any fixed finite number of servers, $2 \leq N < \infty$.
- This holds for general arrival and service distributions (with finite moment bounds).
- The optimality is directly in steady-state.

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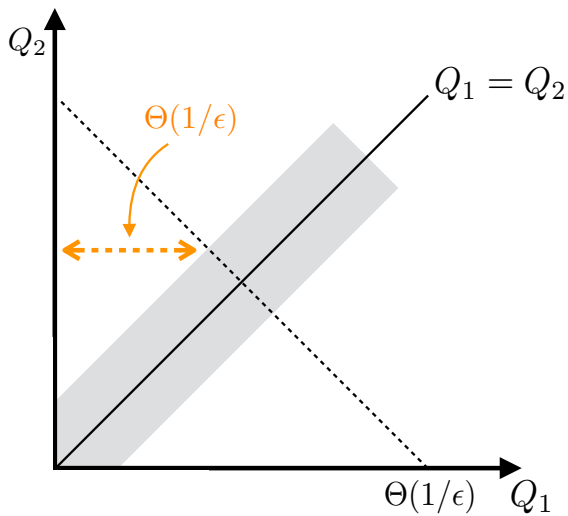
Our methods:

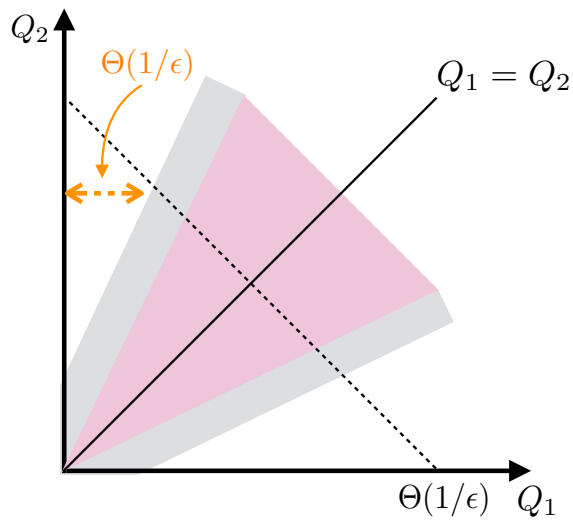
- we use drift-based analysis to establish a new type of state-space collapse.
- we combine this state-space collapse with Chernoff bound.

What is *state-space collapse?*

Informally, it means steady state *'concentrates'* around a subspace.

- the distance between the steady state and a subspace has bounded moment upper bounds.
- the subspace could be a line or a cone in previous works.





Why is *state-space collapse* important?

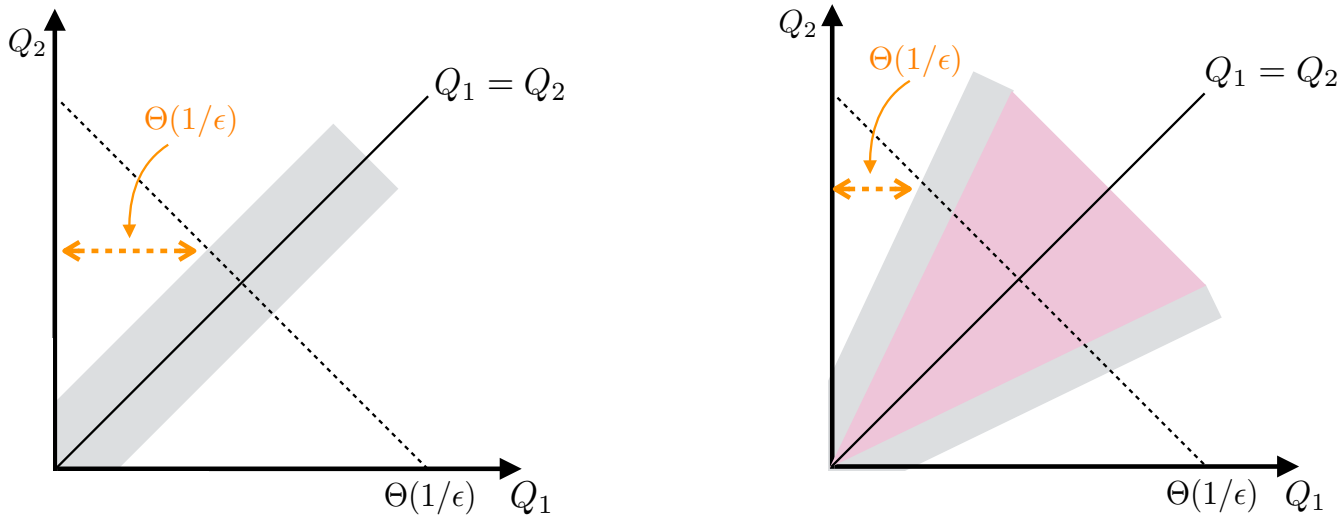
Recall the 'King' equation...

$$\lim_{\epsilon \downarrow 0} \mathbb{E}\left[\left\| \overline{\mathbf{Q}}^{(\epsilon)}(t+1) \right\|_1 \left\| \overline{\mathbf{U}}^{(\epsilon)}(t) \right\|_1 \right] = 0.$$

where the unused service vector $\mathbf{U}(t) = \max\{\mathbf{S}(t) - \mathbf{Q}(t) - \mathbf{A}(t), \mathbf{0}\}$.

- Note that $Q_n(t+1)U_n(t) = 0$ for all n and t.

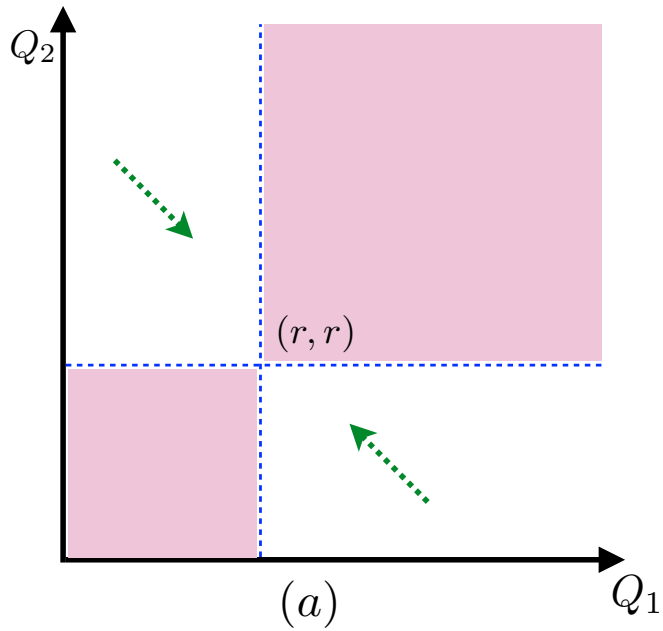
IMPLICATIONS: No server is idle while others with high loads.



In heavy traffic ($\epsilon \to 0$), steady state lies far away from boundary.

What would be the *state-space collapse* in our case?

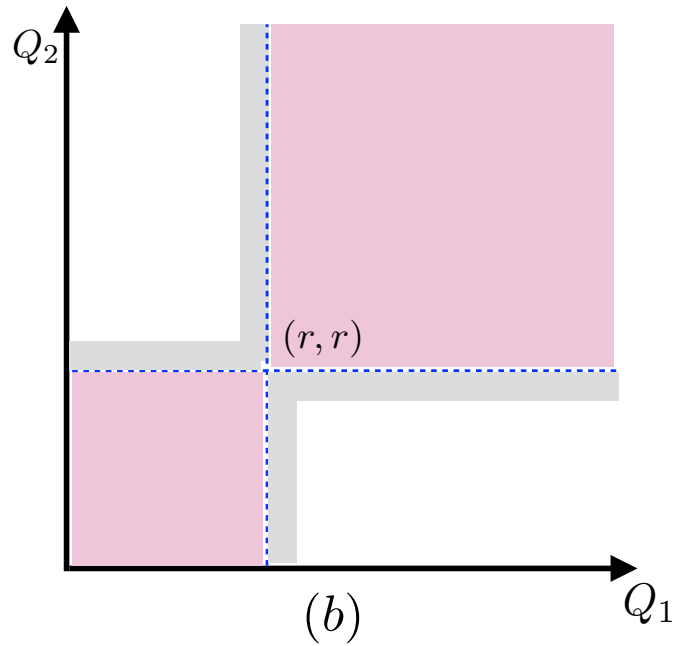
From drift to Chernoff bound...



(a)

Drift towards the pink region due to preference of shorter queues.

From drift to Chernoff bound...

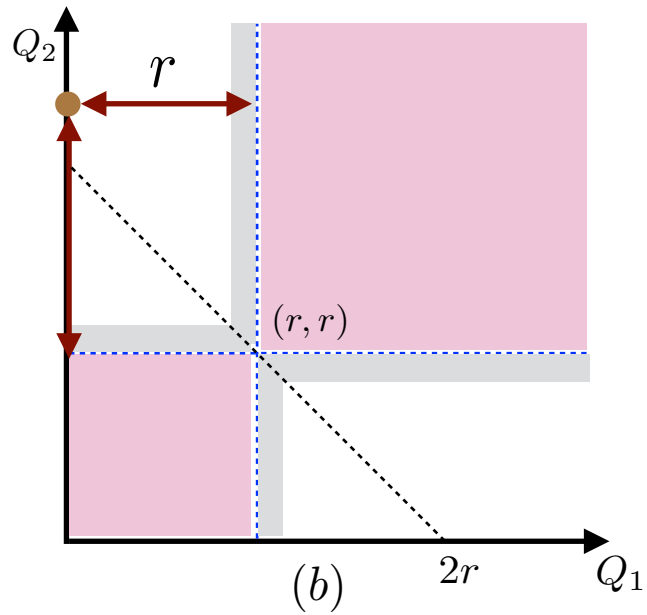


Then, ALL the moments of the distance to pink region are bounded!

$$\mathbb{E}\left[e^{\theta^* d_{\mathcal{R}(r)}\left(\overline{\mathbf{Q}}^{(\epsilon)}\right)}\right] \leq C^*,$$

where both θ^* and C^* are independent of ϵ.

From drift to Chernoff bound...



$$
\begin{aligned}
&\mathbb{E}\left[\overline{Q}_2^{(\epsilon)}(t+1)\overline{U}_1^{(\epsilon)}\right] \\
=&\mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1\mathcal{I}\left(\overline{Q}_2(t+1)\leq 2r, \overline{Q}_1(t+1)=0\right)\right] && (1)\\
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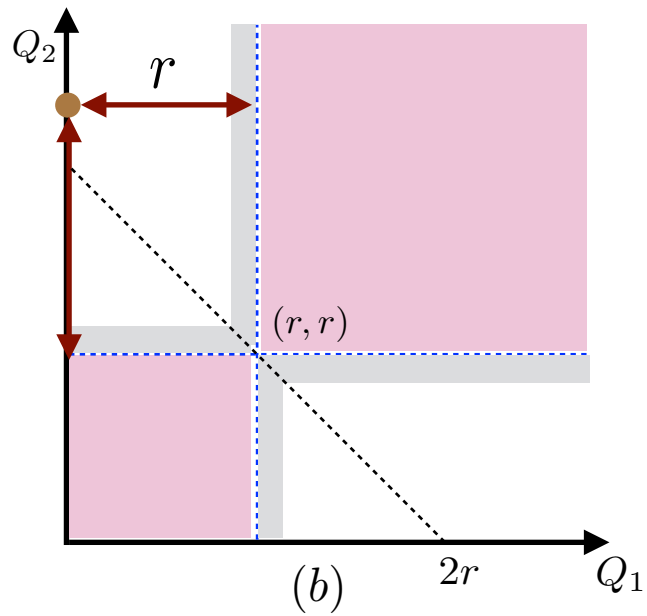

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$$+ \mathbb{E}\left[\overline{Q}_2(t+1)\overline{U}_1\mathcal{I}\left(\overline{Q}_2(t+1) > 2r, \overline{Q}_1(t+1) = 0\right)\right] \quad (2)$$

$$(1) \leq 2r\epsilon, \text{ since } \mathbb{E}\left[\overline{U}_1\right] \leq \epsilon.$$

$$(2) \leq C\frac{1}{\epsilon^2}\mathbb{P}\left(\overline{Q}_2(t+1) > 2r, \overline{Q}_1(t+1) = 0\right) \leq C\frac{1}{\epsilon^2}\frac{1}{e^{\theta r}}$$

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The key requirement: set of servers whose queue lengths are below the threshold are known at the dispatcher

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  - Otherwise, all the new arrivals are dispatched uniformly at random to one of the servers in the system.
- Each server reports its ID to the dispatcher at the end of each time-slot if
  - its queue length is below the threshold
  - AND the dispatcher does not contain its ID (how?)

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Theorem

For any finite T and $d \geq 1$, the policy is throughput and delay optimal in heavy traffic.

Then...is the Logarithmic growth rate also necessary?

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We conjecture so!

Conclusion...

Theorem (Necessary Conditions)

- The threshold r should grow with the traffic load.
- But, it can not grow too fast.
- It provides a sharp characterization of JIQ policy.

Theorem (Sufficient Conditions)

- It is sufficient to have a logarithmic growth rate.
- This resolves a long-standing open problem.
- It provides a useful guideline for practical systems.

Thank you!

Q & A